JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **157**, 568–576 (1991)

# Homoclinic Orbits on Compact Manifolds*

VIERI BENCI AND FABIO GIANNONI

*Istituto di Matematiche Applicate "U. Dini,"*
*Via Bonanno Pisano 25/B, 56126 Pisa, Italy*

*Submitted by Paul H. Rabinowitz*

Received January 10, 1990

Under a suitable assumption on the potential energy $V$, we prove the existence of a homoclinic orbit of the equation $D_t(x'(t)) + \text{grad } V(x(t)) = 0$, $x \in M$, where $M$ is a compact Riemannian manifold.

Our assumption is satisfied, for instance, if the function $V$ has a unique non-degenerate maximum point. 

## 1. INTRODUCTION

In this paper we study the existence of homoclinic orbits for a classical mechanical system constrained on a compact manifold.

Let $\mathcal{M}$ be a regular Riemannian manifold. We consider the differential equation

$$D_t(x'(t)) + \text{grad } V(x(t)) = 0, \tag{1.1}$$

where $V \in C^2(\mathcal{M}, \mathbb{R})$, $x'$ is the derivative of the curve $x(t)$, and $D_t(x')$ is the covariant derivative of $x'$.

(1.2) DEFINITION. A homoclinic orbit is a nonconstant curve solving (1.1) such that

$$\lim_{t \to -\infty} x(t) = \lim_{t \to +\infty} x(t) = x_0$$

$$\lim_{t \to -\infty} x'(t) = \lim_{t \to +\infty} x'(t) = 0.$$

Note that $x_0$ needs to be a critical point of $V$.

In order to state our result of this paper we need the following.

* Work supported by MPI and GNAFA.



0022-247X/91 $3.00

(1.3) *Notation.* If $W$ is a function of class $C^2$ on $\mathcal{M}$, $x \in \mathcal{M}$ and $v \in T_x(\mathcal{M})$.[1] We set

$$H_W(x)[v] = \frac{d^2}{dt^2}\,(W(\gamma(t))|_{t=0},$$

where $\gamma$ is a geodesic on $\mathcal{M}$ (i.e., $D_t(\gamma'(t)) \equiv 0$) defined in a neighbourhood of $t=0$ and such that $\gamma(0)=x$, $\gamma'(0)=v$.

(1.4) THEOREM. *Let $\mathcal{M}$ be a compact Riemannian manifold of class $C^2$, $V \in C^2(\mathcal{M}, \mathbb{R})$, and $x_0$ the only maximum point of $V$. Assume that*

$$\exists \rho_0 > 0 : H_V(x)[v] \leqslant 0 \qquad \forall x : \operatorname{dist}(x, x_0) \leqslant \rho_0, \qquad \forall v \in T_x(\mathcal{M}). \tag{1.5}$$

*Then there exists a homoclinic motion such that*

$$\lim_{t \to -\infty} x(t) = \lim_{t \to +\infty} x(t) = x_0.$$

(1.6) *Remark.* If $V$ has more than one maximum point the result of Theorem (1.4) is in general not true.

For example, let $\mathcal{M}$ be a manifold in $\mathbb{R}^3$ such that

$$B := \{(x_1, x_2, x_3) : x_3 = 0,\, x_1^2 + x_2^2 \leqslant 1\} \subset \mathcal{M},$$

and $V$ a function of class $C^2$ such that $V(x) \leqslant 0$ in $B$ and $V(x) = 0$ in $\mathcal{M} \setminus B$.

Therefore, any homoclinic orbit, if it exists lies in $B$ and satisfies

$$x''(t) = F(x(t)), \qquad \forall t \in \mathbb{R},$$

where $F = -\operatorname{grad} V$.

Now we suppose that

(i) $V$ is radially symmetric,

(ii) 0 is the only critical point of $V$ in $\operatorname{int}(B)$.

By (i) it is easy to see that the angular momentum

$$M(x) = x \times x'$$

is preserved.

We argue indirectly and we suppose that $x(t)$ is a homoclinic orbit such that

$$\lim_{t \to -\infty} x(t) = \lim_{t \to +\infty} x(t) = x_0 \in \partial B.$$

[1] $T_x(\mathcal{M})$ denotes the tangent plane to the manifold $\mathcal{M}$ at $x$.

By Definition (1.2)

$$\lim_{t \to -\infty} x'(t) = \lim_{t \to +\infty} x'(t) = 0,$$

and hence $M(x(t)) = 0$ for every $t \in \mathbb{R}$.

This implies that $x(t)$ lies on a straight line through the origin, so

$$\lim_{t \to +\infty} x(t) = -x_0,$$

and this is a contradiction.

(1.7) *Remark.* The assumption (1.5) is satisfied by a generic potential $V$ since it holds when $x_0$ is nondegenerate. However, we do not know if it can be removed.

(1.8) *Remark.* We recall that recently three other papers appeared on this subject. Rabinowitz in [3] studied the existence of homoclinic orbits on the $N$-dimensional torus and obtained also a multiplicity result.

In [1] Ekeland and Coti Zelati proved the existence of a homoclinic orbit of a nonautonomous periodic Hamiltonian system in $\mathbb{R}^{2N}$.

In [5] Tanaka proved the existence of a homoclinic orbit in $\mathbb{R}^N$, in presence of a singular energy potential $V$ verifying the strong force condition of Gordon and having a unique global maximum.

## 2. The Approximation Scheme

Let $\mathcal{M}$ be a compact Riemannian manifold. By a theorem of Whitney, we can suppose that $\mathcal{M} \subset \mathbb{R}^N$ for $N$ sufficiently large.

Let $x_0 \in \mathcal{M}$. We set

$$\Lambda^1(\mathcal{M}) = \{\gamma \in H^{1,2}(0, 1, \mathbb{R}^N) : \gamma(t) \in \mathcal{M}, \gamma(0) = \gamma(1) = x_0\}.$$

It is well known that $\Lambda^1(\mathcal{M})$ is a Riemannian manifold with tangent plane at $\gamma$ given by

$$T_\gamma \Lambda^1(\mathcal{M}) = \{v \in H^{1,2}(0, 1, \mathbb{R}^N) : v(t) \in T_{\gamma(t)}(\mathcal{M}), \forall t \in [0, 1], v(0) = v(1) = 0\}$$

and scalar product

$$(v, w) \to \int_0^1 \langle v(t), w(t) \rangle_{\gamma(t)} \, dt,$$

where $v, w \in T_\gamma \Lambda^1(\mathcal{M})$ and $\langle \cdot, \cdot \rangle_x$ $(x \in \mathcal{M})$ is the Riemannian structure on $\mathcal{M}$ (cf., e.g., [2]).

Let $V \in C^2(\mathcal{M}, \mathbb{R})$, $x_0$ the only maximum point of $V$, $h = V(x_0)$, and $\varepsilon > 0$. We shall look for critical points for the functional $f_\varepsilon \in C^2(\mathcal{M}, \mathbb{R})$ given by

$$f_\varepsilon(\gamma) = \int_0^1 \frac{1}{2} \langle \gamma', \gamma' \rangle_{\gamma(t)} (h + \varepsilon - V(\gamma))\, dt. \tag{2.1}$$

Note that $\frac{1}{2}\langle \cdot, \cdot \rangle_x (h + \varepsilon - V(x))$ is a Riemannian structure on $\mathcal{M}$.

A critical point for $f_\varepsilon$ is a curve $\gamma \in \Lambda^1(\mathcal{M})$ such that

$$\langle \operatorname{grad} f_\varepsilon(\gamma), v \rangle = 0, \qquad \forall v \in T_\gamma(\mathcal{M}).$$

As it is essentially proved in [2] any critical point of $f_\varepsilon$, $\gamma$, is a curve of class $C^2$ such that $\gamma(t) \in \mathcal{M}$, $\forall t \in [0, 1]$, $\gamma(0) = \gamma(1) = x_0$ and

$$\begin{aligned} D_t(\gamma'(t))(h + \varepsilon - V(\gamma)) - (\langle \operatorname{grad} V(\gamma(t)), \gamma'(t) \rangle_{\gamma(t)} \gamma'(t) \\ + \tfrac{1}{2}\langle \gamma', \gamma' \rangle_{\gamma(t)} \operatorname{grad} V(\gamma(t)) = 0, \qquad \forall t \in [0, 1]. \end{aligned} \tag{2.2}$$

(2.3) *Remark.* Note that if $\gamma$ verifies (2.2) we have

$$\tfrac{1}{2}\langle \gamma', \gamma' \rangle_{\gamma(t)} (h + \varepsilon - V(\gamma)) = c, \qquad \forall t \in [0, 1],$$

and by the definition of $f_{\varepsilon'} c = f_\varepsilon(\gamma)$.

In this section we shall prove the following

(2.4) THEOREM. *Let $x_0$ be the only maximum point for $V$ and $h = V(x_0)$. Then for every $\varepsilon > 0$ there exists $\gamma_\varepsilon \in C^2([0, 1], \mathcal{M})$ such that*

$$\gamma_\varepsilon(0) = \gamma_\varepsilon(1) = x_0$$

$$\gamma_\varepsilon \textit{ verifies } (2.2),$$

$$0 < \delta \leqslant f_\varepsilon(\gamma_\varepsilon) \leqslant M < +\infty,$$

*where $\delta$ and $M$ do not depend on $\varepsilon$ and $f_\varepsilon$ as defined in* (2.1).

*Proof.* Consider

$$c_\varepsilon = \inf_{\operatorname{cat}(A) \geqslant 2} (\sup_A f_\varepsilon), \tag{2.5}$$

where $A$ is a subset of $\Lambda^1(\mathcal{M})$ and $\operatorname{cat}(A)$ denotes the Lusternik and Schnirelman category of $A$ in the space $\Lambda^1(\mathcal{M})$ (see [4]), i.e., the minimal number of closed contractible subset of $\Lambda^1(\mathcal{M})$ which cover $A$.

It is well known that there are compact subset of $\Lambda^1(\mathcal{M})$ such that

$\text{cat}(A)$ is arbitrarily large. Then there exists $M \in \mathbb{R}^+$ (independent of $\varepsilon$) such that

$$c_\varepsilon \leqslant M. \tag{2.6}$$

We claim that there exists $\delta > 0$ independent of $\varepsilon$ such that

$$f_\varepsilon^\delta := \{\gamma \in \Lambda^1(\mathcal{M}) : f_\varepsilon(\gamma) \leqslant \delta\}$$

is contractible on the constant curve $x_0$.

In fact let $r > 0$ be a constant such that the set

$$B(x_0, r) := \{x \in \mathcal{M} : \text{dist}(x, x_0) \leqslant r\}$$

is contractible on $x_0$, and

$$\eta := \inf\{h - V(x) : x \notin B(x_0, r/2)\}$$

which is greater then zero because of our assumptions.

Now if there exists $t \in [0, 1]$ such that $\gamma(t) \notin B(x_0, r/2)$, there exists $t_0 \in [0, t[$ such that $\text{dist}(\gamma(t_0), x_0) = r/2$ and $\gamma(\tau) \notin B(x_0, r/2)$, $\forall \tau \in ]t_0, t]$. Moreover,

$$\text{dist}(\gamma(t), x_0) \leqslant (r/2) + \text{dist}(\gamma(t_0), \gamma(t)), \tag{2.7}$$

while

$$\begin{aligned}
\text{dist}(\gamma(t_0), \gamma(t)) &\leqslant \left(\int_{t_0}^{t} \langle \gamma', \gamma' \rangle_{\gamma(\tau)}\, d\tau\right)^{1/2} \\
&\leqslant \left(\frac{2}{\eta} \int_{t_0}^{t} \frac{1}{2} \langle \gamma', \gamma' \rangle_{\gamma(\tau)} (h + \varepsilon - V(\gamma))\, d\tau\right)^{1/2} \\
&\leqslant \left(\frac{2}{\eta} \int_{0}^{1} \frac{1}{2} \langle \gamma', \gamma' \rangle_{\gamma(\tau)} (h + \varepsilon - V(\gamma))\, d\tau\right)^{1/2} \\
&= \left(\frac{2}{\eta} f_\varepsilon(\gamma)\right)^{1/2}.
\end{aligned}$$

Then, by (2.7), if $\gamma \in f_\varepsilon^\delta$ we have $\text{dist}(\gamma(t), x_0) \leqslant (r/2) + ((2/\eta)\delta)^{1/2}$.

At this point if we have $((2/\eta)\delta)^{1/2} \leqslant (r/2)$, $f_\varepsilon^\delta$ is contractible to the constant curve $x_0$. Therefore if $\text{cat}(A) \geqslant 2$, $\sup_A f_\varepsilon \geqslant \delta$ and

$$c_\varepsilon \geqslant \delta. \tag{2.8}$$

Now since $h + \varepsilon - V(x) \geqslant \varepsilon > 0$ for every $x \in \mathcal{M}$, using standard argument (see, e.g., [2]) it is possible to prove that $f_\varepsilon$ satisfies the Palais–Smale condition (i.e., every sequence $\gamma_n$ such that $f_\varepsilon(\gamma_n) \to c \in \mathbb{R}$ and grad $f_\varepsilon(\gamma_n) \to 0$ has a converging subsequence).

Then $c_\varepsilon$ defined by (2.5) is a critical level for $f_\varepsilon$,[2] and by (2.6) and (2.8) we obtain the proof of (2.4). ∎

## 3. Proof of Theorem (1.4)

Let $V \in C^2(\mathcal{M}, \mathbb{R})$ and $x_0$ be the only maximum point of $V$ and suppose that (1.5) holds. Let $N \in \mathbb{N}$ such that $\mathcal{M} \subset \mathbb{R}^N$, and $|\ |$ the norm in $\mathbb{R}^N$. According to the notation (1.3) if $g(x) = \frac{1}{2}|x - x_0|^2$ we have

$$H_g(x)[v] = |v|^2, \qquad \forall x \in \mathcal{M}, \qquad \forall v \in T_x(\mathcal{M}),$$

so, by (1.5), the function $V_\varepsilon(x) = V(x) - (\varepsilon/2)|x - x_0|^2$ (with $\varepsilon > 0$) is such that

$$H_{V_\varepsilon}(x)[v] \leqslant -\varepsilon |v|^2, \qquad \forall x\colon \mathrm{dist}(x, x_0) \leqslant \rho_0, \qquad \forall v \in T_x(\mathcal{M}). \quad (3.1)$$

Since $x_0$ is the only maximum point for the function $V_\varepsilon$ (with value $h$) by Theorem (2.4) there exists $\gamma_\varepsilon \in C^2([0, 1], \mathcal{M})$ such that

$$\gamma_\varepsilon(0) = \gamma_\varepsilon(1) = x_0, \quad (3.2)$$

$$D_t(\gamma_\varepsilon'(t))(h + \varepsilon - V_\varepsilon(\gamma_\varepsilon(t))) - (\langle \mathrm{grad}\ V_\varepsilon(\gamma_\varepsilon(t)), \gamma_\varepsilon'(t)\rangle_{\gamma_\varepsilon(t)})\, \gamma_\varepsilon'(t)$$
$$+ \tfrac{1}{2}\langle \gamma_\varepsilon', \gamma_\varepsilon'\rangle_{\gamma_\varepsilon(t)}\, \mathrm{grad}\ V_\varepsilon(\gamma_\varepsilon(t)) = 0, \qquad \forall t \in [0, 1], \quad (3.3)$$

$$\tfrac{1}{2}\langle \gamma_\varepsilon', \gamma_\varepsilon'\rangle_{\gamma_\varepsilon(t)}\,(h + \varepsilon - V_\varepsilon(\gamma_\varepsilon)) = c_\varepsilon \in [\delta, M], \quad (3.4)$$

where $0 < \delta \leqslant M < +\infty$ are independent of $\varepsilon \in ]0, 1]$.[3]

Let

$$s_\varepsilon(t) = \int_0^t \frac{\sqrt{c_\varepsilon}}{(h + \varepsilon - V_\varepsilon(\gamma_\varepsilon))}\, d\tau, \qquad \forall t \in [0, 1],$$

$$t_\varepsilon(s) \text{ the inverse of } s_\varepsilon(t),$$

$$x_\varepsilon(s) = \gamma_\varepsilon(t_\varepsilon(s)).$$

It is not difficult to see that $x_\varepsilon$ satisfies the equations

$$\left.\begin{aligned} &D_s(x_\varepsilon'(s)) + \mathrm{grad}\ V_\varepsilon(x_\varepsilon(s)) = 0, \\ &\tfrac{1}{2}\langle x_\varepsilon', x_\varepsilon'\rangle_{x_\varepsilon(s)} = h + \varepsilon - V_\varepsilon(\gamma_\varepsilon), \qquad \forall s \in [0, s_\varepsilon(1)], \end{aligned}\right. \quad (3.5)$$

[2] That is, there exists a critical point $\gamma_\varepsilon$ such that $f_\varepsilon(\gamma_\varepsilon) = c_\varepsilon$.

[3] The estimate for $c_\varepsilon$ independently of $\varepsilon$ can be obtained as in Theorem (2.4), where the case $V_\varepsilon = V$ is considered.

and obviously

$$x_\varepsilon(0) = x_\varepsilon(s_\varepsilon(1)) = x_0.$$

Moreover, by the definition of $s_\varepsilon(t)$ and (3.4), we have

$$\int_0^{s_\varepsilon(1)} (h + \varepsilon - V_\varepsilon(x_\varepsilon))\, ds = \sqrt{c_\varepsilon} \leqslant \sqrt{M}. \tag{3.6}$$

Now observe that the function

$$\varphi_\varepsilon(s) = h + \varepsilon - V_\varepsilon(x_\varepsilon(s))$$

satisfies

$$\varphi''_\varepsilon(s) = -H_{V_\varepsilon}(x_\varepsilon)(x'_\varepsilon, x'_\varepsilon) + \langle \text{grad } V_\varepsilon(x_\varepsilon(s)), \text{grad } V_\varepsilon(x_\varepsilon(s)) \rangle_{x_\varepsilon(s)} > 0$$

if $(x_\varepsilon(s), x'_\varepsilon(s)) \in \{(B(x_0, \rho_0) \times \mathbb{R}^N) \setminus \{(x_0, 0)\}\}$, where $\rho_0$ is defined at (1.5). Moreover by virtue of Theorem (2.4) and the definition of $x_\varepsilon$ we have $x_\varepsilon(s) \not\equiv x_0$, while grad $V_\varepsilon(x_0) = 0$ implies $\varphi'_\varepsilon(0) = \varphi'_\varepsilon(s_\varepsilon(1)) = 0$. Therefore, for every $\rho < \rho_0$ there exist two instants $\alpha_\varepsilon$ and $\beta_\varepsilon$ such that

$$\begin{aligned}
&\text{dist}(x_\varepsilon(\alpha_\varepsilon), x_0) = \text{dist}(x_\varepsilon(\beta_\varepsilon), x_0) = \rho \\
&\text{dist}(x_\varepsilon(\tau), x_0) < \rho, \qquad \forall \tau \in [0, \alpha_\varepsilon[, \\
&\text{dist}(x_\varepsilon(\tau), x_0) < \rho, \qquad \forall \tau \in ]\beta_\varepsilon, s_\varepsilon(1)].
\end{aligned}$$

Then there exists an interval $[a_\varepsilon, b_\varepsilon] \subset [0, s_\varepsilon(1)]$ such that

$$\text{dist}(x_\varepsilon(a_\varepsilon), x_0) = \text{dist}(x_\varepsilon(b_\varepsilon), x_0) = \rho/2, \tag{3.7}$$

$$\text{dist}(x_\varepsilon(\tau), x_0) > \rho/2, \qquad \forall \tau \in [a_\varepsilon, b_\varepsilon]. \tag{3.8}$$

$$\exists \tau_\varepsilon \in [a_\varepsilon, b_\varepsilon] : \text{dist}(x_\varepsilon(\tau_\varepsilon), x_0) = \rho. \tag{3.9}$$

Note that if $S = \sup_{x \in \mathcal{M}} (h + 1 - V(x))$ we have

$$\begin{aligned}
\rho/2 &\leqslant \text{dist}(x_\varepsilon(a_\varepsilon), x_\varepsilon(\tau_\varepsilon)) \\
&\leqslant \left( \int_{a_\varepsilon}^{\tau_\varepsilon} \langle x'_\varepsilon, x'_\varepsilon \rangle_{x(\tau_\varepsilon)}\, d\tau \right)^{1/2} \\
&= \left( \int_{a_\varepsilon}^{\tau_\varepsilon} (h + \varepsilon - V_\varepsilon(x_\varepsilon))\, ds \right)^{1/2} \qquad \text{(by (3.5))} \\
&\leqslant \left( \int_{a_\varepsilon}^{\tau_\varepsilon} S\, d\tau \right)^{1/2} = ((\tau_\varepsilon - a_\varepsilon) S)^{1/2} \leqslant ((b_\varepsilon - a_\varepsilon) S)^{1/2}.
\end{aligned}$$

Then

$$\inf_{\varepsilon} (b_\varepsilon - a_\varepsilon) > 0. \tag{3.10}$$

Now consider the curve

$$\tilde{x}_\varepsilon(t) = \begin{cases} x_\varepsilon(a_\varepsilon), & \forall t < a_\varepsilon \\ x_\varepsilon(t), & \forall t \in [a_\varepsilon, b_\varepsilon]. \\ x_\varepsilon(b_\varepsilon), & \forall t > b_\varepsilon. \end{cases}$$

$\tilde{x}_\varepsilon$ is bounded in $H^{1,2}_{\text{loc}}(\mathbb{R}, \mathbb{R}^N)$, then there exists a sequence $\varepsilon_k \to 0$ such that $\tilde{x}_\varepsilon$ converges in $L^\infty_{\text{loc}}(\mathbb{R}, \mathbb{R}^N)$ to a curve $x_1 \in C^0(\mathbb{R}, \mathscr{M})$.

Now, by (3.6), $b_\varepsilon - a_\varepsilon$ is bounded independently of $\varepsilon$. Then, by (3.10), up to a translation of the parameter $t$, there exists a subsequence $\varepsilon_k^1$ of $\varepsilon_k$ such that $a_{\varepsilon_k^1} \to a_1$, $b_{\varepsilon_k^1} \to b_1$, $b_1 > 0 > a_1$.

In particular

$$x_1 \in C^2(]a_1, b_1[, \mathscr{M}),$$

$$D_t(x_1'(t)) + \text{grad } V(x_1(t)) = 0, \qquad \forall t \in ]a_1, b_1[,$$

$$\tfrac{1}{2}\langle x_1'(t), x_1'(t)\rangle_{x_1(t)} = h - V(x_1(t)), \qquad \forall t \in ]a_1, b_1[,$$

$$\text{dist}(x_1(a_1), x_0) = \text{dist}(x_1(b_1), x_0) = \rho/2,$$

$$\exists \tau_0 \in ]a_1, b_1[ : \text{dist}(x_1(\tau_0), x_0) = \rho,$$

$$\text{dist}(x_1(t), x_0) \geqslant \rho/2, \qquad \forall t \in [a_1, b_1].$$

Analogously, we can prove the existence of an interval $[a_2, b_2] \supset [a_1, b_2]$ and a curve $x_2 \in C^2(]a_2, b_2[, \mathscr{M})$ such that

$$x_2(t) = x_1(t), \qquad \forall t \in [a_1, b_1],$$

$$x_2 \in C^2(]a_2, b_2[, \mathscr{M}),$$

$$D_t(x_2'(t)) + \text{grad } V(x_2(t)) = 0, \qquad \forall t \in ]a_2, b_2[,$$

$$\tfrac{1}{2}\langle x_2'(t), x_2'(t)\rangle_{x_2(t)} = h - V(x_2(t)), \qquad \forall t \in ]a_2, b_2[,$$

$$\text{dist}(x_2(a_2), x_0) = \text{dist}(x_2(b_2), x_0) = \rho/3,$$

$$\text{dist}(x_2(t), x_0) \geqslant \rho/3, \qquad \forall t \in [a_2, b_2],$$

and so on.

Since $\text{dist}(x_k(a_k), x_0) \xrightarrow[k]{} 0$ and $\text{dist}(x_k(b_k), x_0) \xrightarrow[k]{} 0$ by the uniqueness of the Cauchy problem

$$D_t(x'(t)) + \text{grad } V(x(t)) = 0$$

$$x(t_0) = x_0, \qquad x'(t_0) = 0,$$

we obtain $a_k \xrightarrow[k]{} -\infty$ and $b_k \xrightarrow[k]{} +\infty$ and $x_k$ converges in $C^2_{\text{loc}}$ to a curve $x \in C^2(\mathbb{R}, \mathcal{M})$ such that

$$D_t(x'(t)) + \text{grad } V(x(t)) = 0, \qquad \forall t \in \mathbb{R}, \tag{3.11}$$

and

$$\tfrac{1}{2}\langle x', x' \rangle_{x(t)} = h - V(x), \qquad \forall t \in \mathbb{R}. \tag{3.12}$$

Moreover by (3.6) we have

$$\int_{-\infty}^{+\infty} (h - V(x))\, ds < +\infty. \tag{3.13}$$

Then, by virtue of (3.13) and (3.12), using the same arguments of [5, Proposition (3.2)], we see that

$$\lim_{t \to -\infty} x(t) = \lim_{t \to +\infty} x(t) = x_0$$

and

$$\lim_{t \to -\infty} x'(t) = \lim_{t \to +\infty} x'(t) = 0,$$

and this complete the proof of Theorem (1.4). ∎

## ACKNOWLEDGMENT

We thank Professor K. Tanaka for correcting a computational mistake we made in an earlier version.

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